Chapter 7

Multiple-source shortest paths

7.1 Slack costs, relaxed and tense darts, and consistent price vectors

7.1.1 Slack costs

Recall that, for a graph G, we use A_G to denote the dart-vertex incidence matrix.

Let $\boldsymbol{\rho}$ be a vertex vector. The *slack cost vector* with respect to $\boldsymbol{\rho}$ is the vector $\boldsymbol{c}_{\boldsymbol{\rho}} = \boldsymbol{c} - A_G\boldsymbol{\rho}$. That is, the *slack cost of dart d* with respect to $\boldsymbol{\rho}$ is

$$\boldsymbol{c}_{\boldsymbol{\rho}}[d] = \boldsymbol{c}[d] + \boldsymbol{\rho}[\text{tail}(d)] - \boldsymbol{\rho}[\text{head}(d)]$$

In this context, we call $\boldsymbol{\rho}$ a *price vector.*

By a telescoping sum, we obtain

Lemma 7.1.1. *For any path P, $\boldsymbol{c}_{\boldsymbol{\rho}}(P) = \boldsymbol{c}(P) + \boldsymbol{\rho}[start(P)] - \boldsymbol{\rho}[end(P)]$.*

Corollary 7.1.2 (Slack costs preserve optimality)**.** *For fixed vertices s and t, an s-to-t path is shortest with respect to $\boldsymbol{c}_{\boldsymbol{\rho}}$ iff it is shortest with respect to $\boldsymbol{c}$.*

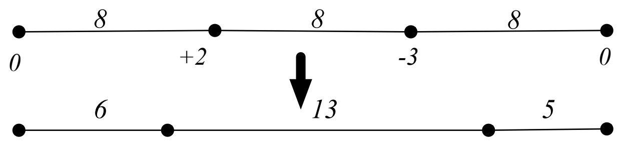


Figure 7.1: This figure shows how a price vector changes lengths of a path. Assume for simplicity that the path's start and end each have price zero. Then the length of the path does not change, despite the changes in the lengths of the darts.

Corollary 7.1.2 is very useful because it alllows us to transform a shortest-path instance with respect to one cost vector, $\boldsymbol{c}$, into a shortest-path instance with respect to another, $\boldsymbol{c}_{\boldsymbol{\rho}}$. This transformation is especially useful if the new costs are nonnegative.

7.1.2 Relaxed and tense darts

Recall from Section 6.1 that a dart d is *relaxed* with respect to $\boldsymbol{c}$ and $\boldsymbol{\rho}$ if

$$\boldsymbol{\rho}[\text{tail}(d)] + \boldsymbol{c}[d] - \boldsymbol{\rho}[\text{head}(d)] \geq 0 \tag{7.1}$$

and *tense* otherwise. Note that the quantity on the left-hand side is the slack cost of d with respect to $\boldsymbol{\rho}$. Thus a dart is relaxed if its slack cost is nonnegative and tense if its slack cost is negative.

A dart d is *tight* if Inequality 7.1 holds with equality, i.e. if its slack cost is zero.

7.1.3 *Consistent* price vectors

A price vector is *consistent* with respect to $\boldsymbol{c}$ if the slack costs of all darts are nonnegative. That is, $\boldsymbol{\rho}$ is a consistent price vector if every dart is relaxed with respect to $\boldsymbol{\rho}$.

Following up on the discussion in Section 7.1.1, a consistent price vector allows us to transform a shortest-path instance with respect to costs some of which are negative into a shortest-path instance in which all costs are nonnegative.

Now we discuss a way to obtain a consistent price vector. For a vertex r, we say that a price vector $\boldsymbol{\rho}$ is the *from-r distance vector* with respect to $\boldsymbol{c}$ if, for every vertex v, $\boldsymbol{\rho}[v]$ is the minimum cost with respect to $\boldsymbol{c}$ of a r-to-v path of darts.

Lemma 7.1.3. *Suppose $\boldsymbol{\rho}$ is the from-r distance vector with respect to $\boldsymbol{c}$ for some vertex r. Then $\boldsymbol{\rho}$ is a consistent price function, and every minimum-cost path starting at r consists of darts that are tight with respect to $\boldsymbol{\rho}$.*

Problem 7.1. *Prove Lemma 7.1.3.*

One motivation for finding a consistent price vector is to transform a shortest-path instance into a simpler shortest-path instance; however, such a transformation seems useless if, as suggested by Lemma 7.1.3, carrying it out requires that we first solve the original shortest-path instance! Nevertheless, the transformation can be quite useful:

- Having distances from one vertex r simplify the computation of distances from other vertices (e.g. the all-pairs-distances algorithm of [Johnson, 1977]).
- An algorithm can maintain a price function and preserve its consistency over many iterations (e.g. the min-cost flow algorithm of [?]).

Problem 7.2.

1. *Show that if P is a minimum-cost u-to-v path w.r.t. costs c, then it is also a minimum-cost u-to-v path w.r.t. slack costs c_ρ. What is the difference between the cost of P w.r.t. c and w.r.t. c_ρ?*

2. *Suppose P_1 and P_0 are both u-to-v paths, where every dart of P_1 is tight (w.r.t. ρ), and where P_0 is a minimum-cost u-to-v path (w.r.t. c). Prove that every dart of P_0 is also tight (w.r.t. ρ).*

3. *Let G be a directed sparse graph with dart lengths (i.e., $|E(G)| = O(|V(G)|)$. Use theproperties of reduced costs to compute the distance between every pair of vertices in G in $O(|V(G)|^2 \log |V(G)|)$ time. This is known as Johnson's algorithm [Johnson, 1977]*

Lemma 7.1.3 suggests a way of using a price vector to certify that a tree is a shortest-path tree.

Lemma 7.1.4. *Let $\boldsymbol{\rho}$ be a price vector that is consistent with respect to $\boldsymbol{c}$. If T is a rooted spanning tree every dart of which is tight then T is a shortest-path tree with respect to $\boldsymbol{c}$.*

Proof. With respect to the slack costs, every dart has nonnegative cost, and every path in T has zero cost, so every path in T is a shortest path with respect to $\boldsymbol{c_\rho}$ and hence (by Corollary 7.1.2) with respect to $\boldsymbol{c}$. □

Lemma 7.1.3 shows that distances form a consistent price function. However, distances do not exist if there are negative-cost cycles. The following lemma states that, in this case, consistent price functions also do not exist.

Lemma 7.1.5. *If $\boldsymbol{\rho}$ is a consistent price vector for G with respect to $\boldsymbol{c}$ then G contains no negative-cost cycles.*

Proof. By a telescoping sum, the slack cost of any cycle C equals its original cost. If $\boldsymbol{\rho}$ is a consistent price vector then every dart of d has nonnegative slack cost, so C has nonnegative slack cost and hence nonnegative cost. □

7.2 Specification of multiple-source shortest paths

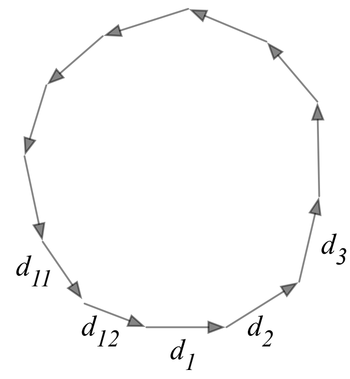
In this chapter, we study a problem called the *multiple-source shortest paths* (MSSP).

- *input:* a directed planar embedded graph G with a designated infinite face f_∞, a vector $\boldsymbol{c}$ assigning lengths to darts, and a shortest-path tree T_0 rooted at one of the vertices of f_∞.

- *output:* a representation of the shortest-path trees rooted at the vertices on the boundary of f_∞.

Our goal is to give an algorithm that solves this problem in $O(n \log n)$ time where n is the number of vertices (assuming no parallel edges). There is an obvious obstacle: each shortest-path tree consists of $n - 1$ edges, so explicitly outputting one for each of the possibly many vertices on the boundary of f_∞ could take far more than $\Omega(n \log n)$ time.

To resolve this difficulty, we use a simple implicit representation of the shortest-path trees. Let $d_1 \cdots d_k$ be the cycle of darts forming the boundary of f_∞



where tail(d_1) is the root of the given shortest-path tree T_0. For $i = 1, \ldots, k$, let T_i be the shortest-path tree rooted at head(d_i). We assume that each shortest-path tree T_i includes only finite-length darts. The algorithm will describe the changes required to transform T_0 into T_1, the changes needed to transform T_1 into T_2, ..., and the changes needed to transform T_{k-1} into T_k. We will show that the total number of changes is at most the number of finite-length darts of G.

7.2.1 Pivots

The basic unit of change in a rooted tree T, called a *pivot*, consists of ejecting one dart d_- and inserting another dart d_+ so that the result is again a rooted tree. A pivot is specified by the pair (d_-, d_+) of darts.[1]

Transforming T from T_{i-1} to T_i consists of

- a *special* pivot that ejects the dart whose head is head(d_i), and inserts the dart rev(d_i) (now T is head(d_i)-rooted), and
- a sequence of *ordinary* pivots each of which ejects a dart d' and inserts a dart $\hat{d}$ with the same head.

For $i = 1, \ldots, k$, the MSSP algorithm outputs the sequence of pivots that transform T_{i-1} into T_i. We shall show that the amortized time per pivot is $O(\log n)$. The number of special pivots is k. In the next section, we show that the number of ordinary pivots is at most the number of finite-length darts. It follows that the running time is $O(n \log n)$.

[1]The term *pivot* comes from an analogy to the network-simplex algorithm.

7.3 Contiguity property of shortest-path trees in planar graphs

In this section we prove a contiguity property for the trees T_i that is crucial in bounding the number of steps of the MSSP algorithm. To facilitate the exposition we first consider the non-degenerate case, where shortest paths are unique. We analyze the degenerate case in section **??**. The same arguments are used in the analysis of the single-source max-flow algorithm presented in Chapter 10.

Let G be a graph, let $\boldsymbol{c}$ be a dart vector, and let d be a dart. For a vertex u, we say d is *u-tight with respect to $\boldsymbol{c}$* if d is tight with respect to the from-u distance vector with respect to $\boldsymbol{c}$.

Lemma 7.3.1. *There is a u-rooted shortest-path tree containing d iff d is u-tight.*

Problem 7.3. *Prove Lemma 7.3.1 using Lemma 7.1.4.*

Note that our unique shortest-paths assumption implies that a dart is u-tight if and only if is in the shortest path tree rooted at u.

Lemma 7.3.2 (Consecutive-Roots Lemma)**.** *Let G be a planar embedded graph with infinite face f_∞, and let $\boldsymbol{c}$ be a dart vector. Let $(d_1\ d_2\ \cdots)$ be the cycle of darts forming the boundary of f_∞.*

For each dart d, the set

$$\{i\ :\ d \text{ is in } T_i\} \tag{7.2}$$

forms a consecutive subsequence of the cycle $(1\ 2\ \cdots)$.

Proof. Let v be a vertex. Let T' be the v-rooted shortest-path tree using darts in the reverse direction. (Formally, we use the costs c' defined by $c'[d] = c[rev(d)]$). For each vertex r on f_∞, if d is the last dart in the shortest r-to-v path, then the first dart in the v-to-r path in T' is $\text{rev}(d)$. That is, r is a descendent of $\text{tail}(d)$ in T'. Since the tree T' does not cross itself, the set $\{r_i \in f_\infty : r_i \in T'_{\text{tail}(d)}\}$ is a consecutive subsequence of the boundary of f_∞. The lemma follows. □

Corollary 7.3.3. *The number of pivots required to transform shortest-path tree T_{i-1} into T_i, summed over all i, is at most the number of finite-length darts.*

7.4 The abstract MSSP algorithm

We present an abstract description of an algorithm for MSSP. Later we will present a more detailed description.

```
def MSSP($G, f_\infty, T$):
    pre: $T$ is a shortest-path tree rooted at a vertex of $f_\infty$
    let $(d_1\ d_2\ \cdots\ d_s)$ be the darts of $f_\infty$, where $\text{tail}_G(d_1)$ is the root of $T$
    for $i := 1, 2, \ldots, s$,
    $T$ is a tail($d_i$)-rooted shortest-path tree
are not $(A_i, B_i) :=$ CHANGEROOT$(T, d_i)$    transform $T$ to a head($d_i$)-rooted
                shortest-path tree by removing dart-set $A_i$ and adding $B_i$
    the darts of $A_i$ are not head($d_i$)-tight
  return $(A_1, \ldots, A_s)$ and $(B_1, \ldots, B_s)$
```

The algorithm calls the subroutine CHANGEROOT, for each dart d_i of f_∞ in order. CHANGEROOT transforms T_{i-1} into T_i. It returns two sets of darts; the set A_i of darts in T_{i-1} but not in T_i, and the set B_i of darts in T_i but not in T_{i-1}.

7.4.1 Analysis of the abstract algorithm

Lemma 7.4.1. *For each dart d, there is at most one iteration i such that $d \in A_i$.*

Proof. By the Consecutive-Roots Lemma (Corollary 7.3.2), there is at most one integer i such that d is in T_i but is not in T_{i+1}. □

Corollary 7.4.2. *$\sum_i |A_i|$ is at most the number of darts.*

7.5 CHANGEROOT: the inner loop of the MSSP algorithm

The procedure CHANGEROOT(T, d_i) called in MSSP consists of a loop, each iteration of which selects a dart to add to T and a dart to remove from T. We show later that setting up the loop takes $O(\log n)$ time, the number of iterations is $|A_i|$, and each iteration takes amortized $O(\log n)$ time. The time for CHANGEROOT is therefore $O((|A_i|+1)\log n)$. Summing over all i and using Corollary 7.4.2, we infer that the total time for CHANGEROOT and therefore for MSSP is $O(n \log n)$.

tree T^*, enables the vertex v

In each iteration, the algorithm determines that a dart $\hat{d}$ not in T must be added to T. It then *pivots $\hat{d}$ into T*, which means

- removing from T the dart whose head is head$(\hat{d})$ and
- adding $\hat{d}$ to T.

This operation is called a *pivot* because it resembles a step of the network-simplex algorithm.

Conceptually, CHANGEROOT is as follows. Let c_0 denote the tail(d_i)-to-head(d_i) distance in G. To initialize, CHANGEROOT temporarily sets the cost of the dart rev(d_i) to $-c_0$. It then performs a special pivot, removing from T the dart whose head is head(d_i), and inserting rev(d_i) into T. It then gradually increases the cost of rev(d_i) to its original cost while performing ordinary pivots as necessary to maintain that T is a shortest-path tree.

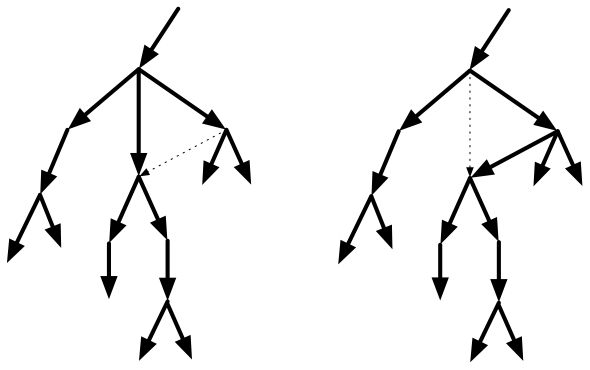


Figure 7.2: The two trees differ by a single pivot.

After the initialization, the procedure uses a variable t to represent the current modified cost of $\text{rev}(d_i)$. The costs of other darts remain unmodified. Let $\boldsymbol{c}_t$ denote the vector of costs. That is,

$$\boldsymbol{c}_t[d] = \begin{cases} t & \text{if } d = \text{rev}(d_i) \\ \boldsymbol{c}[d] & \text{otherwise} \end{cases}$$

Under these costs, the cost of the path in T to a vertex v is denoted $\boldsymbol{\rho}_{T,t}[v]$. The vector $\boldsymbol{\rho}_{T,t}$ is not represented explictly by the algorithm; we use it in the analysis and proof of correctness.

After initialization, the procedure ChangeRoot repeats the following two steps until t equals the original cost of $\text{rev}(d_i)$:

- Increase t until any further increase would result in some dart d becoming tense with respect to $\boldsymbol{c}_t$ and $\boldsymbol{\rho}_{T,t}$.
- Pivot d into T.

Lemma 7.5.1. *Throughout the execution, T is a shortest-path tree with respect to the costs $\boldsymbol{c}_t$.*

Proof. We prove the lemma by induction. For the basis of the induction, we must prove that T is a shortest-path tree immediately after the special pivot, when $\text{rev}(d_i)$ is pivoted in.

Define $\boldsymbol{\rho}$ to be the from-tail(d_i) distance vector with respect to $\boldsymbol{c}$. At the very beginning of ChangeRoot, before $\text{rev}(d_i)$ is pivoted in, T is a shortest-path tree, so by Lemma 7.1.3 its darts are tight with respect to $\boldsymbol{\rho}$ and $\boldsymbol{c}$. Since $\boldsymbol{c}_t$ is identical to $\boldsymbol{c}$ on these darts, they are also tight with respect to $\boldsymbol{c}_t$.

By Lemma 7.1.4, it remains only to show that $\text{rev}(d_i)$ itself is tight when it is first pivoted in. At this time, its cost $\boldsymbol{c}_t[\text{rev}(d_i)]$ is

$$-c_0 = -(\boldsymbol{\rho}[\text{tail}(\text{rev}(d_i))] - \boldsymbol{\rho}[\text{head}(\text{rev}(d_i))])$$

so its slack cost is zero. This completes the basis of the induction.

Now for the induction step. We assume T is a shortest-path tree. The variable t is increased until further increase would result in some dart d becoming tense with respect to $\boldsymbol{c}_t$ and $\boldsymbol{\rho}_{T,t}$. At this point, d is tight. Therefore, by Lemma 7.1.4, pivoting d into T yields a shortest-path tree. □

7.6 Which darts are candidates for pivoting in?

In this section, we consider the loop of ChangeRoot in which t is increased.

The algorithm pivots in a dart d only if necessary, i.e. if continuing the shrinking or growing phase without pivoting in d would result in T not being a shortest-path tree, in particular if d would become tense with respect to $\boldsymbol{c}$ and $\boldsymbol{\rho}_{T,t}$. A dart d is in danger of becoming tense only if its slack cost with respect to $\boldsymbol{\rho}_{T,t}$ is decreasing.

We define a labeling of the vertices with colors. For each vertex v, if the root-to-v path contains $\text{rev}(d_i)$ then we say v is *blue*, and otherwise we say v is *red*.

Lemma 7.6.1. *Suppose that t increases by Δ. For a dart d not in T, the slack cost of d*

- *decreases if $\text{tail}_G(d)$ is red and $\text{head}_G(d)$ is blue,*
- *increases if $\text{tail}_G(d)$ is blue and $\text{head}_G(d)$ is red, and*
- *otherwise does not change.*

$\text{rev}(d_i)$ Observation **??**.

Suppose $\text{rev}(d_i)$ is in T. Then the fundamental cut of $\text{rev}(d_i)$ with respect to T has the form $\vec{\delta}_G(S)$ where S is the set of red vertices. When the cost of $\text{rev}(d_i)$ increases by Δ, by Lemma 7.6.1, the darts of this cut (not including $\text{rev}(d_i)$) undergo a decrease of Δ in their slack costs. By Fundamental-Cut/Fundamental-Cycle Duality (4.6.1), these are the darts belonging to the the fundamental cycle of $\text{rev}(d_i)$ in the dual G^* with respect to T^*.

Since $\text{tail}_{G^*}(\text{rev}(d_i))$ is f_∞, this cycle, minus the dart $\text{rev}(d_i)$ itself, is the path in T^* from $\text{head}_{G^*}(\text{rev}(d_i))$ to f_∞. We summarize this result as follows:

Lemma 7.6.2. *Suppose $rev(d_i)$ is in T and its cost increases by Δ. This results in a decrease by Δ in the slack costs of the darts of the $head_{G^*}(rev(d_i))$-to-f_∞ path in T^*, and a decrease by Δ in the slack costs of the reverse darts.*

7.7 Efficient implementation

The concrete version of MSSP does not explicitly represent the distances $\boldsymbol{\rho}_{T,t}[\cdot]$. Instead, it represents the slack costs of the darts whose edges are not in T. This permits an efficient implementation of ChangeRoot.

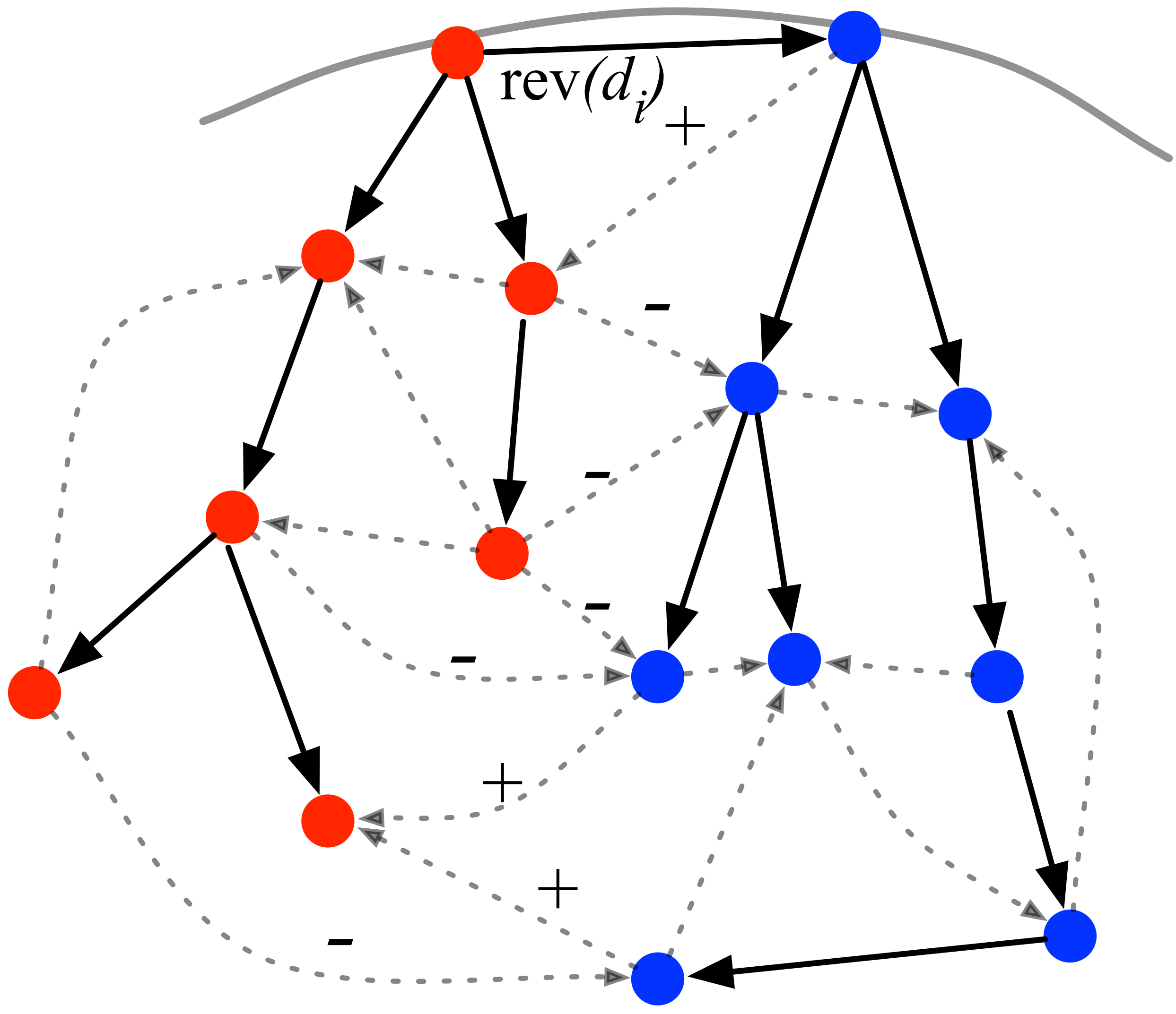


Figure 7.3: This figure shows the coloring of vertices. Those vertices reached through rev(d_i) are blue and the others are red. The red-to-blue nontree arcs are labeled with **-** to indicate that their slack costs decrease as the algorithm progresses. The blue-to-red nontree arcs are labeled with + to indicate that their slack costs increase. The red-to-red and blue-to-blue arcs remain unchanged.

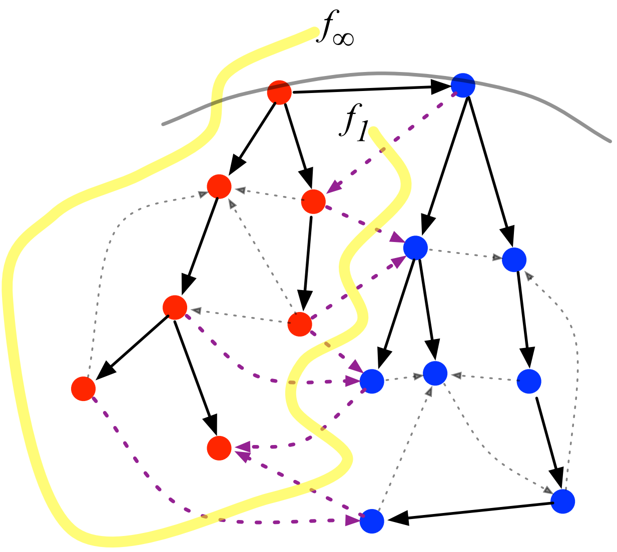


Figure 7.4: This figure illustrates that the edges whose slack costs are changing form a cycle, the fundamental cycle of rev(d_i). The duals of darts on the f_1-to-f_∞ path decrease in slack cost, and their reverses increase in slack cost.

7.7.1 ChangeRoot

Now we give the procedure ChangeRoot. (We later describe how it is implemented using data structures.)

```
def ChangeRoot($T, d_i$):
     pre: The root of $T$ is tail$_G(d_i)$
1    initialize $A_i, B_i := \emptyset$.
2    $t := -\boldsymbol{c}[d_i] +$ slack cost of $d_i$
3    remove from $T$ the dart whose head is head$_G(d_i)$ and add rev$(d_i)$
4    comment: now the root of $T$ is head$(d_i)$
5    repeat
6        let $P$ be the head$_{G^*}$(rev$(d_i)$)-to-$f_\infty$ path in $T^*$
7        find a dart $\hat{d}$ in $P$ whose slack cost $s$ is minimum
8        $\Delta = \min\{s, c[\text{rev}(d_i)] - t\}$
9        subtract $\Delta$ from the slack costs of darts in $P$
10       add $\Delta$ to $t$ and to the slack costs of reverses of darts in $P$
12       if $\Delta < c[\text{rev}(d_i)] - t$
11           remove from $T$ the dart whose head is head$_G(\hat{d})$, and add it to $A_i$
12           add $\hat{d}$ to $T$ and to $B_i$
13   until $t = c[\text{rev}(d_i)]$
14   return $(A_i, B_i)$
```

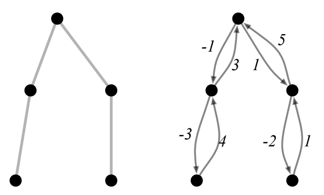


Figure 7.5: The left figure shows a tree. The right figure indicates that the tree must be represented by a data structure in such a way that each edge has two weights, the slack cost of the rootward dart and the slack cost of the leafward dart. The operation AddToAncestor(v, Δ) operates on the edges of the v-to-root path. For each edge, the operation subtracts Δ from the rootward dart and adds Δ to the leafward dart.

7.7.2 Data structure

To support efficient implementation, MSSP represents T in two ways. It represents T directly via a table, and represents T indirectly by representing the interdigitating tree T^* (rooted at f_∞) by a link-cut tree.

The table parentD$[\cdot]$ stores, for each vertex v that is not the root of T, the dart of T whose head is v (the parent dart of v).

The link-cut tree representing T^* has a node for each vertex of G^* and a node for each edge of T^*. The edge-nodes are assigned pairs of weights; the weights associated with edge e are $(w_R(e), w_L(e))$, where w_R is the slack cost of the dart of e oriented towards the root f_∞ and w_L is the slack cost of the dart of e oriented away from the root.

In each iteration, in Line 7 the algorithm uses a AncestorFindMin operation on the link-cut tree representing T^* to select the dart $\hat{d}$ to insert into T and obtain the dart's slack cost Δ. In Line 11 the algorithm uses the table parentD$[\cdot]$ to find which dart must therefore be removed from T. In Lines 9 and 10, the algorithm updates the slack costs along P using the operation AddToAncestors$(\text{head}_{G^*}(\text{rev}(d_i)), \Delta)$.

The topological changes to T in Lines 11 and 12 are carried out by making topological changes to T^* using operations Cut, Evert, and Link. First, the dart $\hat{d}$ being pivoted into T must be removed from T^* by a Cut operation. This breaks T^* into two trees, one rooted at f_∞ and one rooted at $\text{tail}_{G^*}(\hat{d})$. Next, as shown in Figure 7.6, an Evert operation must be performed on the latter tree to reroot it at $\text{head}_{G^*}(d')$ where $d' = \text{parentD}[\text{head}_G(\hat{d})]$ is the dart to be removed from T in Line 11. This reorients the path from $\text{head}_{G^*}(d')$ to $\text{head}_{G^*}(\hat{d})$. Finally, now that $\text{head}_{G^*}(d')$ is the root, this tree is linked to the one rooted at f_∞ by performing Link$(\text{head}_{G^*}(d'), \text{tail}_{G^*}(d'))$.

The link-cut tree must support GetWeight, Evert, AddToAncestor, and AncestorFindMin. Moreover, since eversion changes which dart of an edge e is oriented towards the root, the weights must be handled carefully.

Each iteration of the repeat-loop of Line 5 thus requires a constant number of link-cut-tree operations It follows that the time for iteration i of MSSP requires

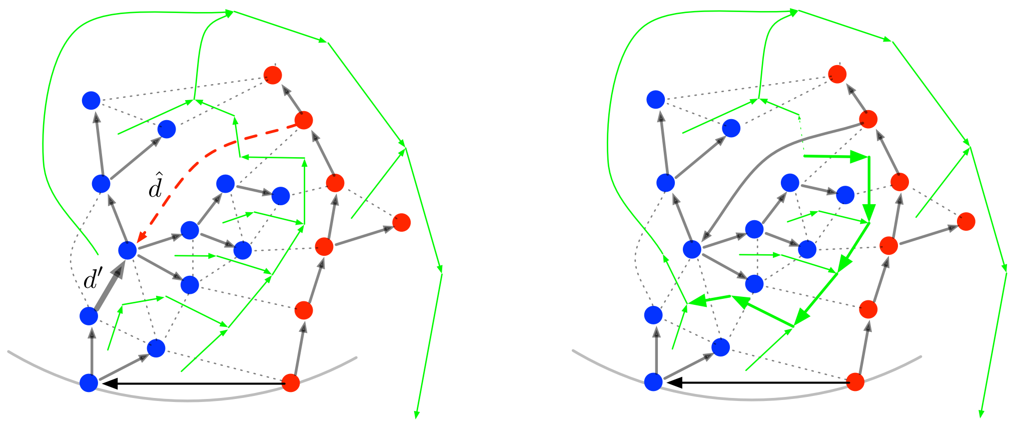


Figure 7.6: This figure shows the need for reversing the direction of a path in the dual tree T^*. The dual tree is shown using rootward darts. The diagram on the left shows the situation just before a pivot. The red dashed edge $\hat{d}$ is about to be pivoted into T, which causes the corresponding dual edge to be removed from T^*. The thick black edge must be removed from T, which causes the corresponding dual edge to be added to T^*. The figure on the right shows the resulting situation. Note that the thick edges in the dual tree have reversed direction.

amortized time $O((1 + |A_i|) \log n)$. By Corollary 7.4.2, $\sum_i (1 + |A_i|)$ is at most the number of darts plus the size of the boundary of f_∞. Since the initialization of the data structures takes linear time, it follows that MSSP requires time $O(n \log n)$.

7.8 Number of pivots - the degenerate case

In this section we show that it is not necessary to assume unique shortest paths. Note that we had used this assumption only to bound the number of pivots in Section 7.3. For the correctness we only used the fact that the dart $\hat{d}$ chosen in line 7 of ChangeRoot is a dart with minimum slack cost on the path P, so that T remains a shortest path tree at all times. When shortest paths are not unique, there may be more than one dart with minimum slack cost in P. We show that if $\hat{d}$ is chosen to be the leafmost such dart (i.e., the one farthest from f_∞ on P), then each dart is ejected from T at most once during the entire execution of the algorithm, and hence the number of pivots is bounded by the number of darts in the graph.

Lemma 7.8.1. *Let v be any vertex. For $j = 0, 1, 2, \ldots\ldots$, let P_j^v be the root-to-v path in the shortest-path tree T after j pivots. The paths $P_0^v, P_1^v, \ldots$ are mutually noncrossing.*

Proof. Assume the theorem is not true. Let j be the minimum integer such that, for some vertex v and some integer $i < j$, P_j^v crosses P_i^v. The path P_j^v is obtained from P_{j-1}^v by a pivot. The pivot must be ordinary since otherwise, because the dart $\text{rev}(\hat{d})$ that pivots into the shortest path tree lies on $f\infty$, any path that crosses P_j^v also crosses P_{j-1}^v, contradicting the definition of j. Let T be the shortest-path tree just before the j'th pivot, and let xy be the dart that enters T in the j'th pivot. Then P_{j-1}^v is the path to v in T. Write $P_{j-1}^v = Q_1 \circ Q_2$ where the end of Q_1 (and start of Q_2) is y. Let R be the path in T to x. Then $P_j^v = R \circ xy \circ Q_2$. Since Q_2 belongs to P_{j-1}^v, the choice of j implies that P_i^v does not cross Q_2.

Consider the cycle $C = R \circ xy \circ \text{rev}(Q_1)$. Observe that for any dart d of C other than xy, either d or $\text{rev}(d)$ belongs to T. Thus, for any dart d that is enclosed by C such that $d \in T^*$, d is a descendant of xy in T^*. j'th pivot. Let o be a dummy vertex embedded inside f_∞ connected by edges to all the vertices of f_∞. Let s be the first vertex of P_i^v. Let $S = os \circ P_i^v$.

Suppose S and C cross. Consider the shortest prefix S' of S that crosses C. Since o is not enclosed by C, the last dart of S' is strictly enclosed by C. Let z be the tail of the last dart of S'. Let u be the first vertex of S after z that belongs to C, or v if S does not intersect C after z. The subpath $S[z, u]$ of S is a shortest path, so all of its darts have slack zero. By choice of $S[z, u]$, it is enclosed by C. Hence, if at least one the darts of $S[z, u]$ is not in T, then it is a proper descendant in T^* of xy. In this case the leafmost rule does not select xy to pivot in, which is a contradiction. It follows that $S[z, u]$ must be a path in T.

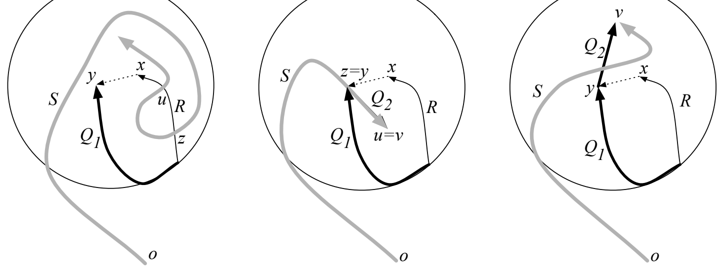


Figure 7.7: Illustrations for the proof of Lemma 7.8.1.

We now consider three different options for u. If $u \in Q_1$ then $Q_1[u]$ and $S[z, u]$ are distinct subpaths of T ending at u, which contradicts T being a tree. Similarly, if $u \in R$ then $R[u]$ and $S[z, u]$ are distinct subpaths of T ending at u, which contradicts T being a tree. See Figure 7.7 (left).

The only remaining option is $u = v$, and since $S[z, u]$ is a path in T whose only vertex on C is z, then it must also be that $z = y$ and $S[z, u] = Q_2$. But then S does not cross $R \circ xy \circ Q_2 = P_j^v$. See Figure 7.7 (middle).

To conclude, either S crosses C, but not P_j^v, or S does not cross C. In the former case P_i^v does not cross P_j^v because P_i^v is a subpath of S. In the latter case, any crossing of S and P_j^v is also a crossing of S and P_{j-1}^v, contradicting the choice of j. See Figure 7.7 (middle). □

Lemma 7.8.2. *Once a dart pivots out of T it does not pivot back in within the same call to* ChangeRoot.

Proof. Consider an iteration of the repeat-loop. Let $\hat{d}$ be the dart selected in Line 7 to be inserted into T, and let $\tilde{d}$ be the dart that in Line 11 is removed from T. $\hat{d}$ and $\tilde{d}$ have the same head v. Since the dart $\hat{d}$ selected in Line 7 is in P, its head v is blue and its tail is red. Consider the moment just after Lines 11 and 12 are executed. Since $\hat{d}$ has been added to T, v becomes red. Since along the execution of ChangeRoot vertices only change color from blue to red, no dart whose head is v will be inserted into T in the remainder of the execution of ChangeRoot. □

Lemma 7.8.3. *Each dart is ejected from T at most once during the entire execution of the algorithm*

Proof. Let d be a dart. Assume d is ejected from T twice during the execution of the algorithm. By Lemma 7.8.2, it must do so in two distinct calls to ChangeRoot. Let $i < j$ be such that d is ejected during the calls to ChangeRoot on T_i and T_j. Then $d \in T_i$, $d \notin T_{i+1}$, $d \in T_j$, $d \notin T_{j+1}$. Therefore, two of the root-to-head(d) paths in the trees $T_i, T_{i+1}, T_j, T_{j+1}$ must cross, contradicting Lemma 7.8.1. □

7.9 Using the output of the MSSP algorithm

There are several ways to use the output of the MSSP algorithm.

7.9.1 Paths

We can use the output to build a data structure that supports queries of the form ShortestPath(head(d_i), v) where d_i is a dart of the boundary of f_∞ and v is an arbitrary vertex.

Fix a vertex v, and let $b_1, \ldots, b_{\text{degree}(v)}$ be the darts entering v. For each dart d_i on the boundary of f_∞, define

$$g(i) = \min\{j : b_j \text{ is head}(d_i)\text{-tight}\}$$

Note that $b_{g(i)}$ is the last dart in a shortest head(d_i)-to-v path.

Corollary 7.3.2 implies that, for each entering dart b_j, the set $\{i : g(v,i) = j\}$ forms a consecutive subsequence of $(1\ 2\ \cdots)$. For different entering darts, the subsequences are disjoint. Using, e.g., the data structure of [Mehlhorn and Naher, 1990], one can represent $g(\cdot)$ in $O(\text{degree}(v))$ space so that computing $g(i)$ takes $O(\log\log k)$ time where k is the number of darts on the boundary of f_∞. Over all vertices v, the space required is linear in the size of the graph. Given these data structures, a query ShortestPath(head(d_i), v) can be answered iteratively by constructing the path backwards from v to head(d_i), one dart per iteration. The time for each iteration is $O(\log\log k)$, so constructing a shortest-path consisting of ℓ edges takes time $O(\ell \log\log n)$.

7.9.2 Distances

We describe a Distances algorithm that processes the output of the MSSP algorithm to answer a given set of q queries of the form distance(tail(d_i), v) in time $O((n+q)\log n)$. The Distances algorithm maintains a link-cut tree representation of T as T goes from T_0 to T_1 to ... to T_k. The link-cut tree assigns weights to the vertices. The Distances algorithm ensures that the weight of v is the length of the root-to-v path in T. For $i = 0, 1, \ldots, k-1$, when T is the tail(d_i)-rooted shortest-path tree, the Distances algorithm queries the link-cut tree to find the weights of those vertices v for which the tail(d_i)-to-v distance is desired. The time per pivot and per query is $O(\log n)$.

Now we give the algorithm more formally. We represent the evolving shortest-path tree T using a link-cut-tree data structure that supports AddToDescendants and GetWeight. We maintain the invariant that the weight assigned in T to each vertex v is the length of the root-to-v path in T.

```
def DISTANCES($d_1 \cdots d_k, T_0, \mathcal{P}, \mathcal{Q}$):
      pre: $d_1 \cdots d_k$ are the darts of $f_\infty$,
            $T_0$ is the tail($d_1$)-rooted shortest-path tree,
            $\mathcal{P}$ is the sequence of pivots $(d', \hat{d})$ produced by the MSSP algorithm
            $\mathcal{Q}$ is an array such that $\mathcal{Q}[i]$ is a set of vertices
      post: returns the set $\mathcal{D} = \{(i, v, dist(\text{tail}(d_i), v)) : v \in Q[i]\}$ of distances
1     initialize $T$ :=link-cut tree representing $T_0$ with weight($v$) := $\boldsymbol{c}(T_0[v])$)
2     initialize $\mathcal{D} = \emptyset$, $i := 0$
3     for each pivot $(d', \hat{d})$ in $\mathcal{P}$,
4        if $(d', \hat{d})$ is a special pivot, // about to transform to next tree...
5           $i := i + 1$
            // ... but first find distances in current tree
6           for each $v \in Q[i]$,
7              append $(i, v, T.$GETWEIGHT$(v))$ to $\mathcal{D}$
         // perform pivot
8        $T.$CUT(tail($d'$)) // remove $d'$ from $T$
9        $T.$ADDTODESCENDANTS(head($d'$), $-T.$GETWEIGHT(head($d$)))
10       $T.$LINK(tail($\hat{d}$), head($\hat{d}$)) // add $\hat{d}$ to $T$
11       $T.$ADDTODESCENDANTS(head($d'$), $\boldsymbol{c}[\hat{d}] + T.$GETWEIGHT(tail($d$)))
12    return $\mathcal{Q}$
```

In Line 4, if the next pivot in the sequence is a special pivot, it means that the current tree is a shortest-path tree, so now is the time to find distances from the current root. The algorithm maintains the invariant that the weight of each vertex v in T is the length of the root-to-v path in T, so in Line 7 the distance to v is the weight of v.

Lines 8-11 carry out a pivot. Line 8 ejects d', breaking the tree into two trees, one with the same root as before and one rooted at head(d'). Line 9 updates the weights of vertices in the latter tree to preserve the invariant. Line 10 inserts $\hat{d}$, forming a single tree once again. Line 11 again updates the weights of the vertices that were in the latter tree to preserve the invariant.

Problem 7.4. *(minimum st-cut in undirected planar graphs using MSSP.)*

1. *Let G be an n-vertex planar graph with dart lengths. Let f and g be two faces, and P be a simple path from a vertex on f to a vertex on g. Give an $O(n \log n)$-time algorithm that returns a minimum-length simple cycle that encloses exactly one of f and g, and crosses P at most once.*
   Hint: the concept of trimming along a path (Section 4.9) is useful here.

2. *Let G be an undirected connected planar graph with non-negative edge lengths. Let f and g be two faces of G. Give an $O(n \log n)$-time algorithm that finds the shortest cycle in G that separates f and g.*

3. *For vertices s, t in an* undirected *graph with edge capacities, a minimum st-cut is a set of edges that separates s and t and whose sum of capacities is minimum. Let s, t be two distinct vertices in an undirected connected planar graph G with non-negative edge capacities. Give an $O(n \log n)$-time algorithm that returns a minimum st-cut in G.*

Problem 7.5. *(substring longest common subsequence using MSSP.) A string is a sequence of characters. The longest common subsequence (LCS) of two strings S and T is the longest string W that can be obtained from both S and T by deleting characters. For example, The longest common subsequence of "abccababca" and "badbadbad" is "bababa". There is an easy, well known dynamic programming algorithm that computes the LCS in $O(|S||T|)$ time. Consider the following grid graph $G = (V, E)$. The vertex set V is $\{v_{ij} : 0 \leq i \leq |S|, 0 \leq j \leq$. Vertex v_{ij} is adjacent to vertices $v_{(i-1)j}, v_{(i+1)j}, v_{i(j-1)}, v_{i(j+1)}$. So far this is just a regular grid graph. In addition, if $S[i] = T[j]$ then $v_{(i-1)(j-1)}v_{ij}$ is an edge in E.*

7.9.3 Distance data structure

Let $\mathcal{D}$ be a dynamic data structure. That is, a data structure that undergoes updates over time. At any point during the execution of an algorithm, the data structure $\mathcal{D}$ can answer queries according to the current version of the data structure, or be modified by changing a a pointer or some other value stored by the data structure.

We say that $\mathcal{D}$ is an *ephemeral* dynamic data structure if an update overwtires the old version of $\mathcal{D}$, leaving only the new version of $\mathcal{D}$ available for queries and further updates. We say that $\mathcal{D}$ is persistent if one can query any version of the data structure. Driscoll et. al [Driscoll et al., 1989] showed a general technique to make any ephemeral data structure persistent. If the in-degree of the data structure (i.e., the number of pointers pointing to each node) is bounded by a constant, then the space and time cost to represent each update operation is amortized constant, and the cost of each query operation becomes slower by a constant factor in the worst case.

We apply this technique to the ephemeral link-cut tree data structure representing the evolving shortest-path tree T. Since there are $O(n)$ pivots during the entire algorithm, and since each pivot can be implemented in $O(\log n)$ time, the size of the resulting persistent representation of T is $O(n \log n)$, and the time to construct it remains $O(n \log n)$. Now, for any dart d_i of f_∞, and any vertex v, the distance $dist(\text{tail}(d_i), v))$ can be returned, by quering the weight of v in the version of T just before the special pivot in which d_{i+1} enters T.

7.10 Chapter Notes

The MSSP algorithm was developed by Klein [Klein, 2005]. Cabello, Chambers and Erickson [Cabello and Chambers, 2007, Cabello et al., 2012] presented a similar algorithm, and generalized it to graphs embedded on surfaces of higher genus. The algorithm presented in this Chapter is based on the variant of Cabello et al. [Cabello et al., 2012].

Eisenstat and Klein [Eisenstat and Klein, 2013] showed that in unweighted planar graphs the algorithm can be implemented in $O(n)$ time witnhout link-cut tree, using only parent pointers to represent the primal and dual trees. They also showed a matching sorting-based lower bound of $\Omega(n \log n)$ time for any algorithm for the MSSP problem that only uses comparison and addition of edge-weights. That is, no such algorithm can compute the sequence of pivots computed by the MSSP algorithm in $o(n \log n)$ time.

The MSSP algorithm in this chapter is closely related to the maximum *st*-flow algorithm in Chapter 10.